

# Systems of the Kowalevski type and discriminantly separable polynomials

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## Abstract

Starting from the notion of discriminantly separable polynomials of degree two in each of three variables, we construct a class of integrable dynamical systems. These systems can be integrated explicitly in genus two theta-functions in a procedure which is similar to the classical one for the Kowalevski top. The discriminantly separable polynomials play the role of the Kowalevski fundamental equation. The natural examples include the Sokolov systems and the Jurdjevic elasticae.

A partir de la notion des polynômes avec les discriminants séparables qui sont du second degré par rapport chacune de trois variables, nous construisons une classe des systèmes dynamiques. On peut résoudre explicitement ces systèmes via les fonctions thêta de genre 2, par une procédure similaire cette classique pour la toupie de Kowalevski. Les polynômes avec les discriminants séparables jouent le rôle de l'équation fondamentale de Kowalevski. Les exemples naturels incluent les systèmes de Sokolov et les élastiques de Jurdjevic.

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<sup>1</sup>.

## 1 Introduction

### 1.1 A short note on discriminantly separable polynomials

The Kowalevski top [20] is one of the most celebrated integrable systems. There is a waste literature dedicated to understanding of Kowalevski original integration procedure, to its modern versions and hidden symmetries (see for example [21], [19], [13], [14], [22], [1], [6], [15], [12], [2], [4]).

In a recent paper [8] of one of the authors of the present paper, a new approach to the Kowalevski integration procedure has been suggested. The novelty has been based on a new notion introduced therein of *discriminantly separable polynomials*. A family of such polynomials has been constructed there as pencil equations from the theory of conics

$$\mathcal{F}(w, x_1, x_2) = 0,$$

where  $w, x_1, x_2$  are the pencil parameter and the Darboux coordinates respectively. (For classical applications of the Darboux coordinates see Darboux's book [5], for modern applications see the book [9] and [7].) The key algebraic property of the pencil equation, as quadratic equation in each of three variables  $w, x_1, x_2$  is: *all three of its discriminants are expressed as products of two*

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polynomials in one variable each:

$$\begin{aligned}\mathcal{D}_w(\mathcal{F})(x_1, x_2) &= P(x_1)P(x_2) \\ \mathcal{D}_{x_1}(\mathcal{F})(w, x_2) &= J(w)P(x_2) \\ \mathcal{D}_{x_2}(\mathcal{F})(w, x_1) &= P(x_1)J(w)\end{aligned}\tag{1}$$

where  $J, P$  are polynomials of degree 3 and 4 respectively, and the elliptic curves

$$\Gamma_1 : y^2 = P(x), \quad \Gamma_2 : y^2 = J(s)$$

are isomorphic (see Proposition 1 of [8]).

In the so-called *fundamental Kowalevski equation* (see formula (10) below, and also [20], [19], [13])  $Q(w, x_1, x_2) = 0$ , the polynomial  $Q(w, x_1, x_2)$  appears to be an example of a member of the family, as it was shown in [8] (Theorem 3). Moreover, all main steps of the Kowalevski integration now follow as easy and transparent logical consequences of the theory of discriminantly separable polynomials. Let us mention here just one relation, see Corollary 1 from [8] (known in the context of the Kowalevski top as *the Kowalevski magic change of variables*):

$$\begin{aligned}\frac{dx_1}{\sqrt{P(x_1)}} + \frac{dx_2}{\sqrt{P(x_2)}} &= \frac{dw_1}{\sqrt{J(w_1)}} \\ \frac{dx_1}{\sqrt{P(x_1)}} - \frac{dx_2}{\sqrt{P(x_2)}} &= \frac{dw_2}{\sqrt{J(w_2)}}.\end{aligned}\tag{2}$$

There is a natural and important question in this context:

*Are there other integrable dynamical systems related to discriminantly separable polynomials?*

Referring to this question, we have already constructed discrete integrable systems related to discriminantly separable polynomials which are associated to quad-graphs in [11]. Now we are constructing a new class of integrable continuous systems, generalizing the Kowalevski top. Thus we call the members of that class – *systems of the Kowalevski type*. A relationship with the discriminantly separable polynomials gives us possibility to perform an effective integration procedure, and to provide an explicit integration formulae in the theta-functions, in general, associated with genus two curves, as in the original case of Kowalevski. The first examples of such systems have been constructed in [10]. Let us point out here one very important moment regarding the Kowalevski top and all the systems of Kowalevski type: the main issue in integration procedures is related to the elliptic curves  $\Gamma_1, \Gamma_2$  and the two-valued groups related to these elliptic curves, although, as we know, the final part of integration of the Kowalevski is related to a genus two curve. This, in a sense unexpected, surprising or possibly not clarified enough jump in genus from 1 to 2 is now explained in our Theorem 1, and it becomes a trade mark of all systems of the Kowalevski type.

## 1.2 An overview of the paper

The paper is organized as follows. In the Section 2 we introduce *systems of the Kowalevski type* of differential equations by generalizing Kowalevski's considerations. Subsection 2.2 presents the Sokolov system from [23] and [18] as an example of systems of the Kowalevski type. As a result, we are providing a full explanation of its integration procedure. In the third subsection we construct a new example of such system and in the fourth subsection we give integration procedure in terms of genus two theta functions and related to them  $P_i, P_{ij}$  functions. Following generalized Kötter transformation from [8], here we reformulate it for polynomials of degree three. That gives us possibility to integrate systems in two ways, using properties of  $\wp$ -function and using generalized Kötter transformation. The subsection 2.5 presents one more method for obtaining systems of Kowalevski type by studying first integrals and invariant relations. The Kowalevski top may be seen as a special subcase, and it serves as a principle motivating example.

In the Section 3 we consider a simple deformation of Kowalevski case by using simplest linear gauge transformation on the Kowalevski fundamental equation. As the outcome we get *the Jurdjevic elasticae* from [15]. Systems analogue to the Jurdjevic elasticae have been obtained before by Komarov and Kuznetsov (see [16] and [17]), and they also serve as motivating examples for our study of systems of Kowalevski type. We give here the explicit solutions of all these problems in terms of  $P_i, P_{ij}$  functions. Another system of Kowalevski type is constructed in the Section 2.6 following [20] and by use of a sequence of skilful tricks and identities.

Since the main part of this paper is motivated by the Kowalevski top and by the Kowalevski integration procedure, and since it is the milestone of the classical integrable systems, we find it useful to extract some of the key moments of the Kowalevski work [20], here.

## 1.3 Fundamental steps in the Kowalevski integration procedure

Let us recall briefly that the Kowalevski top [20] is a heavy spinning top rotating about a fixed point, under the conditions  $I_1 = I_2 = 2I_3, I_3 = 1, y_0 = z_0 = 0$ . Here  $(I_1, I_2, I_3)$  denote the principal moments of inertia,  $(x_0, y_0, z_0)$  is the center of mass,  $c = Mgx_0$ ,  $M$  is the mass of the top,  $(p, q, r)$  is the vector of angular velocity and  $(\gamma_1, \gamma_2, \gamma_3)$  are cosines of the angles between  $z$ -axis of the fixed coordinate system and the axes of the coordinate system that is attached to the top and whose origin coincides with the fixed point. Then the equations of motion take the following form, see [20], [13]:

$$\begin{aligned}
2\dot{p} &= qr & \dot{\gamma}_1 &= r\gamma_2 - q\gamma_3 \\
2\dot{q} &= -pr - c\gamma_3 & \dot{\gamma}_2 &= p\gamma_3 - r\gamma_1 \\
\dot{r} &= c\gamma_2 & \dot{\gamma}_3 &= q\gamma_1 - p\gamma_2.
\end{aligned} \tag{3}$$

System (3) has three well known integrals of motion and a fourth integral discovered by Kowalevski

$$\begin{aligned}
2(p^2 + q^2) + r^2 &= 2c\gamma_1 + 6l_1 \\
2(p\gamma_1 + q\gamma_2) + r\gamma_3 &= 2l \\
\gamma_1^2 + \gamma_2^2 + \gamma_3^2 &= 1 \\
((p + iq)^2 + \gamma_1 + i\gamma_2)((p - iq)^2 + \gamma_1 - i\gamma_2) &= k^2.
\end{aligned} \tag{4}$$

After the change of variables

$$\begin{aligned}
x_1 &= p + iq, & e_1 &= x_1^2 + c(\gamma_1 + i\gamma_2) \\
x_2 &= p - iq, & e_2 &= x_2^2 + c(\gamma_1 - i\gamma_2)
\end{aligned} \tag{5}$$

the first integrals (4) transform into

$$\begin{aligned}
r^2 &= E + e_1 + e_2 \\
rc\gamma_3 &= G - x_2e_1 - x_1e_2 \\
c^2\gamma_3^2 &= F + x_2^2e_1 + x_1^2e_2 \\
e_1e_2 &= k^2,
\end{aligned} \tag{6}$$

with  $E = 6l_1 - (x_1 + x_2)^2$ ,  $F = 2cl + x_1x_2(x_1 + x_2)$ ,  $G = c^2 - k^2 - x_1^2x_2^2$ . From the first integrals, one gets

$$(E + e_1 + e_2)(F + x_2^2e_1 + x_1^2e_2) - (G - x_2e_1 - x_1e_2)^2 = 0$$

which can be rewritten in the form

$$e_1P(x_2) + e_2P(x_1) + R_1(x_1, x_2) + k^2(x_1 - x_2)^2 = 0 \tag{7}$$

where the polynomial  $P$  is

$$P(x_i) = x_i^2E + 2x_1F + G = -x_i^4 + 6l_1x_i^2 + 4l cx_i + c^2 - k^2, \quad i = 1, 2$$

and

$$\begin{aligned}
R_1(x_1, x_2) &= EG - F^2 \\
&= -6l_1x_1^2x_2^2 - (c^2 - k^2)(x_1 + x_2)^2 - 4lc(x_1 + x_2)x_1x_2 + 6l_1(c^2 - k^2) - 4l^2c^2.
\end{aligned}$$

Note that  $P$  from the formula above depends only on one variable, which is not obvious from its definition. Denote

$$R(x_1, x_2) = Ex_1x_2 + F(x_1 + x_2) + G.$$

From (7), Kowalevski gets

$$(\sqrt{P(x_1)e_2} \pm \sqrt{P(x_2)e_1})^2 = -(x_1 - x_2)^2 k^2 \pm 2k\sqrt{P(x_1)P(x_2)} - R_1(x_1, x_2). \quad (8)$$

After a few transformations, (8) can be written in the form

$$\left[ \sqrt{e_1} \frac{\sqrt{P(x_2)}}{x_1 - x_2} \pm \sqrt{e_2} \frac{\sqrt{P(x_1)}}{x_1 - x_2} \right]^2 = (w_1 \pm k)(w_2 \mp k), \quad (9)$$

where  $w_1, w_2$  are the solutions of an equation, quadratic in  $s$ :

$$Q(s, x_1, x_2) = (x_1 - x_2)^2 s^2 - 2R(x_1, x_2)s - R_1(x_1, x_2) = 0. \quad (10)$$

The quadratic equation (10) is known as **the Kowalevski fundamental equation**. The discriminant separability condition for  $Q(s, x_1, x_2)$  is satisfied

$$\mathcal{D}_s(Q)(x_1, x_2) = 4P(x_1)P(x_2)$$

$$\mathcal{D}_{x_1}(Q)(s, x_2) = -8J(s)P(x_2), \quad \mathcal{D}_{x_2}(Q)(s, x_1) = -8J(s)P(x_1)$$

with

$$J(s) = s^3 + 3l_1 s^2 + s(c^2 - k^2) + 3l_1(c^2 - k^2) - 2l^2 c^2.$$

The equations of motion (3) can be rewritten in new variables  $(x_1, x_2, e_1, e_2, r, \gamma_3)$  in the form:

$$\begin{aligned} 2\dot{x}_1 &= -if_1, & \dot{e}_1 &= -me_1 \\ 2\dot{x}_2 &= if_2, & \dot{e}_2 &= me_2. \end{aligned} \quad (11)$$

There are two additional differential equations for  $\dot{r}$  and  $\dot{\gamma}_3$ . Here  $m = ir$  and  $f_1 = rx_1 + c\gamma_3$ ,  $f_2 = rx_2 + c\gamma_3$ . One can easily check that

$$f_1^2 = P(x_1) + e_1(x_1 - x_2)^2, \quad f_2^2 = P(x_2) + e_2(x_1 - x_2)^2. \quad (12)$$

Further integration procedure is described in [20], and in the Subsection 2.4, we are going to develop analogue techniques for more general systems in details.

## 2 Systems of the Kowalevski type

### 2.1 Systems of the Kowalevski type. Definition

Now, we are going to introduce a class of dynamical systems, which generalize the Kowalevski top. Instead of the Kowalevski fundamental equation (see formula (10)), we start here from an arbitrary discriminantly separable polynomial of degree two in each of three variables.

Given a discriminantly separable polynomial of the second degree in each of three variables

$$\mathcal{F}(x_1, x_2, s) := A(x_1, x_2)s^2 + B(x_1, x_2)s + C(x_1, x_2), \quad (13)$$

such that

$$\mathcal{D}_s(\mathcal{F})(x_1, x_2) = B^2 - 4AC = 4P(x_1)P(x_2),$$

and

$$\begin{aligned} \mathcal{D}_{x_1}(\mathcal{F})(s, x_2) &= 4P(x_2)J(s) \\ \mathcal{D}_{x_2}(\mathcal{F})(s, x_1) &= 4P(x_1)J(s). \end{aligned}$$

Suppose, that a given system in variables  $x_1, x_2, e_1, e_2, r, \gamma_3$ , after some transformations reduces to

$$\begin{aligned} 2\dot{x}_1 &= -if_1, & \dot{e}_1 &= -me_1, \\ 2\dot{x}_2 &= if_2, & \dot{e}_2 &= me_2, \end{aligned} \quad (14)$$

where

$$f_1^2 = P(x_1) + e_1A(x_1, x_2), \quad f_2^2 = P(x_2) + e_2A(x_1, x_2). \quad (15)$$

Suppose additionally, that the first integrals and invariant relations of the initial system reduce to a relation

$$P(x_2)e_1 + P(x_1)e_2 = C(x_1, x_2) - e_1e_2A(x_1, x_2). \quad (16)$$

The equations for  $\dot{r}$  and  $\dot{\gamma}_3$  are not specified for the moment and  $m$  is a function of system's variables.

If a system satisfies the above assumptions we will call it *a system of the Kowalevski type*. As it has been pointed out in the Introduction, see formulae (7, 10, 11, 12), the Kowalevski top is an example of the systems of the Kowalevski type.

The following theorem is quite general, and concerns all the systems of the Kowalevski type. It explains in full a subtle mechanism of a quite miraculous jump in genus, from one to two, in integration procedure, which has been observed in the Kowalevski top, and now it is going to be established as a characteristic property of the whole new class of systems.

**Theorem 1** *Given a system which reduces to (14, 15, 16). Then the system is linearized on the Jacobian of the curve*

$$y^2 = J(z)(z - k)(z + k),$$

where  $J$  is a polynomial factor of the discriminant of  $\mathcal{F}$  as a polynomial in  $x_1$  and  $k$  is a constant such that

$$e_1e_2 = k^2.$$

*Proof.* Indeed, from the equations of motion on  $e_i$  we get

$$e_1 e_2 = k^2,$$

with some constant  $k$ . Now, we get

$$\left( \sqrt{e_1} \sqrt{P(x_2)} \pm \sqrt{e_2} \sqrt{P(x_1)} \right)^2 = C(x_1, x_2) - k^2 A(x_1, x_2) \pm 2\sqrt{P(x_1)P(x_2)}k.$$

From the last relations, we get

$$\left( \sqrt{e_1} \sqrt{\frac{P(x_2)}{A}} + \sqrt{e_2} \sqrt{\frac{P(x_1)}{A}} \right)^2 = (s_1 + k)(s_2 - k)$$

and

$$\left( \sqrt{e_1} \sqrt{\frac{P(x_2)}{A}} - \sqrt{e_2} \sqrt{\frac{P(x_1)}{A}} \right)^2 = (s_1 - k)(s_2 + k),$$

where  $s_1, s_2$  are the solutions of the quadratic equation  $\mathcal{F}(x_1, x_2, s) = 0$  in  $s$ . From the last equations we get

$$\begin{aligned} 2\sqrt{e_1} \sqrt{\frac{P(x_2)}{A}} &= \sqrt{(s_1 + k)(s_2 - k)} + \sqrt{(s_1 - k)(s_2 + k)} \\ 2\sqrt{e_2} \sqrt{\frac{P(x_1)}{A}} &= \sqrt{(s_1 + k)(s_2 - k)} - \sqrt{(s_1 - k)(s_2 + k)}. \end{aligned}$$

Since  $s_i$  are solutions of the quadratic equation  $F(x_1, x_2, s_i) = 0$ , using Viète formulae and discriminant separability condition, we get

$$s_1 + s_2 = -\frac{B}{A}, \quad s_2 - s_1 = \frac{\sqrt{4P(x_1)P(x_2)}}{A}. \quad (17)$$

From the last equation, we get

$$(s_1 - s_2)^2 = 4 \frac{P(x_1)P(x_2)}{A^2}.$$

Using the last equation, we have

$$\begin{aligned} f_1^2 &= \frac{P(x_1)}{(s_1 - s_2)^2} \left[ (s_1 - s_2)^2 + 4e_1 \frac{P(x_2)}{A^2} A \right] \\ &= \frac{P(x_1)}{(s_1 - s_2)^2} \left[ (s_1 - s_2)^2 + \left( \sqrt{(s_1 - k)(s_2 + k)} + \sqrt{(s_1 + k)(s_2 - k)} \right)^2 \right] \\ &= \frac{P(x_1)}{(s_1 - s_2)^2} \left[ \sqrt{(s_1 - k)(s_1 + k)} + \sqrt{(s_2 + k)(s_2 - k)} \right]^2. \end{aligned}$$

Similarly

$$f_2^2 = \frac{P(x_2)}{(s_1 - s_2)^2} \left[ \sqrt{(s_1 - k)(s_1 + k)} - \sqrt{(s_2 + k)(s_2 - k)} \right]^2.$$



From the last two equations and from the equations of motion, we get

$$\begin{aligned} 4\dot{x}_1^2 &= -\frac{P(x_1)}{(s_1 - s_2)^2} \left[ \sqrt{s_1^2 - k^2} + \sqrt{s_2^2 - k^2} \right]^2 \\ 4\dot{x}_2^2 &= -\frac{P(x_2)}{(s_1 - s_2)^2} \left[ \sqrt{s_1^2 - k^2} - \sqrt{s_2^2 - k^2} \right]^2, \end{aligned}$$

and then

$$\begin{aligned} \frac{dx_1}{\sqrt{P(x_1)}} + \frac{dx_2}{\sqrt{P(x_2)}} &= -i \frac{\sqrt{(s_1 - k)(s_1 + k)}}{s_1 - s_2} dt \\ \frac{dx_1}{\sqrt{P(x_1)}} - \frac{dx_2}{\sqrt{P(x_2)}} &= -i \frac{\sqrt{(s_2 - k)(s_2 + k)}}{s_1 - s_2} dt. \end{aligned}$$

From the discriminant separability, one gets (see Corollary 1 from [8]):

$$\begin{aligned} \frac{dx_1}{\sqrt{P(x_1)}} + \frac{dx_2}{\sqrt{P(x_2)}} &= \frac{ds_1}{\sqrt{J(s_1)}} \\ -\frac{dx_1}{\sqrt{P(x_1)}} + \frac{dx_2}{\sqrt{P(x_2)}} &= \frac{ds_2}{\sqrt{J(s_2)}} \end{aligned} \quad (18)$$

and finally

$$\begin{aligned} \frac{ds_1}{\sqrt{\Phi(s_1)}} + \frac{ds_2}{\sqrt{\Phi(s_2)}} &= 0 \\ \frac{s_1 ds_1}{\sqrt{\Phi(s_1)}} + \frac{s_2 ds_2}{\sqrt{\Phi(s_2)}} &= -idt, \end{aligned} \quad (19)$$

where

$$\Phi(s) = J(s)(s - k)(s + k),$$

where  $\Phi$  is a polynomial of degree up to six.

Thus, relations (19) define the Abel map on a curve  $y^2 = \Phi(s)$ , which has genus 2 if the roots of the polynomial  $\Phi$  are distinct.  $\square$

The last Theorem basically formalizes the original considerations of Kowalevski, in a slightly more general context of the discriminantly separable polynomials.

We are going to present below the Sokolov system [23] as an example of a system of the Kowalevski type, and to provide one new example of the systems of the Kowalevski type.

## 2.2 Example: Sokolov system as a system of the Kowalevski type

Sokolov in [23], [24] considered the Hamiltonian

$$\hat{H} = M_1^2 + M_2^2 + 2M_3^2 + 2c_1\gamma_1 + 2c_2(\gamma_2 M_3 - \gamma_3 M_2) \quad (20)$$

on  $e(3)$  with the Lie-Poisson brackets

$$\{M_i, M_j\} = \epsilon_{ijk} M_k, \quad \{M_i, \gamma_j\} = \epsilon_{ijk} \gamma_k, \quad \{\gamma_i, \gamma_j\} = 0 \quad (21)$$

where  $\epsilon_{ijk}$  is the totally skew-symmetric tensor. In this section we will prove that this system belongs to the class of systems of the Kowalevski type.

The Lie-Poisson bracket (21) has two well-known Casimir functions

$$\begin{aligned} \gamma_1^2 + \gamma_2^2 + \gamma_3^2 &= a, \\ \gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3 &= b. \end{aligned}$$

Following [18] and [20] we introduce the new variables

$$z_1 = M_1 + iM_2, \quad z_2 = M_1 - iM_2$$

and

$$\begin{aligned} e_1 &= z_1^2 - 2c_1(\gamma_1 + i\gamma_2) - c_2^2 a - c_2(2\gamma_2 M_3 - 2\gamma_3 M_2 + 2i(\gamma_3 M_1 - \gamma_1 M_3)), \\ e_2 &= z_2^2 - 2c_1(\gamma_1 - i\gamma_2) - c_2^2 a - c_2(2\gamma_2 M_3 - 2\gamma_3 M_2 + 2i(\gamma_1 M_3 - \gamma_3 M_1)). \end{aligned}$$

The second integral of motion for the system (20) may be rewritten as

$$e_1 e_2 = k^2. \quad (22)$$

The equations of motion in the new variables  $z_i, e_i$  can be written in the form of (14) and (15), and this corresponds to the definition of the systems of the Kowalevski type. It is easy to show that:

$$\dot{e}_1 = -4iM_3 e_1, \quad \dot{e}_2 = 4iM_3 e_2$$

and

$$\begin{aligned} -\dot{z}_1^2 &= P(z_1) + e_1(z_1 - z_2)^2, \\ -\dot{z}_2^2 &= P(z_2) + e_2(z_1 - z_2)^2 \end{aligned} \quad (23)$$

where  $P$  is a polynomial of fourth degree given by

$$P(z) = -z^4 + 2Hz^2 - 8c_1bz - k^2 + 4ac_1^2 - 2c_2^2(2b^2 - Ha) + c_2^4 a. \quad (24)$$

In order to demonstrate that the Sokolov system belongs to the class of the systems of the Kowalevski type, we still need to show that a relation of the form (16) is satisfied and we have to relate it with certain discriminantly separable polynomial of the form of (13).

Starting from the equations

$$\dot{z}_1 = -2M_3(M_1 - iM_2) + 2c_2(\gamma_1 M_2 - \gamma_2 M_1) + 2c_1 \gamma_3$$

and

$$\dot{z}_2 = -2M_3(M_1 + iM_2) + 2c_2(\gamma_1 M_2 - \gamma_2 M_1) + 2c_1 \gamma_3$$

one can get the following

**Lemma 1** *The product of the derivatives of the variables  $z_i$  is:*

$$\dot{z}_1 \cdot \dot{z}_2 = - \left( F(z_1, z_2) + (H + c_2^2 a)(z_1 - z_2)^2 \right), \quad (25)$$

where  $F(z_1, z_2)$  is given by:

$$F(z_1, z_2) = -\frac{1}{2} \left( P(z_1) + P(z_2) + (z_1^2 - z_2^2)^2 \right). \quad (26)$$

After equating the square of  $\dot{z}_1 \dot{z}_2$  from the relation (25) with product  $\dot{z}_1^2 \cdot \dot{z}_2^2$  with  $\dot{z}_i^2$ ,  $i = 1, 2$ , from (23) we get

**Lemma 2** *The variables  $z_1, z_2, e_1, e_2$  of the Sokolov system satisfy the following identity:*

$$\begin{aligned} & (z_1 - z_2)^2 [2F(z_1, z_2)(H + c_2^2 a) + (z_1 - z_2^2)^4 (H + c_2^2 a)^2 - P(z_1)e_2 - P(z_2)e_1 \\ & - e_1 e_2 (z_1 - z_2)^2] + F^2(z_1, z_2) - P(z_1)P(z_2) = 0. \end{aligned} \quad (27)$$

Denote by  $C(z_1, z_2)$  a biquadratic polynomial such that

$$F^2(z_1, z_2) - P(z_1)P(z_2) = (z_1 - z_2)^2 C(z_1, z_2).$$

**Proposition 1** *The Sokolov system is a system of the Kowalevski type. It can be explicitly integrated in the theta-functions of genus 2.*

*Proof.* We can rewrite relation (27) in the form:

$$P(z_1)e_2 + P(z_2)e_1 = \tilde{C}(z_1, z_2) - e_1 e_2 (z_1 - z_2)^2, \quad (28)$$

with

$$\tilde{C}(z_1, z_2) = C(z_1, z_2) + 2F(z_1, z_2)(H + c_2^2 a) + (H + c_2^2 a)^2 (z_1 - z_2)^2. \quad (29)$$

Further integration procedure may be done following Theorem 1, since the Sokolov system satisfies all the assumptions of the systems of the Kowalevski type: (27), (28) and (23). The discriminantly separable polynomial of three variables of degree two in each of them which plays the role of the Kowalevski fundamental equation in this case is

$$\tilde{F}(z_1, z_2, s) = (z_1 - z_2)^2 s^2 + \tilde{B}(z_1, z_2)s + \tilde{C}(z_1, z_2) \quad (30)$$

with

$$\tilde{B}(z_1, z_2) = F(z_1, z_2) + (H + c_2^2 a)(z_1 - z_2)^2.$$

The discriminants of (30) as polynomials in  $s$  and in  $z_i$ , for  $i = 1, 2$  are

$$\mathcal{D}_s(\tilde{F})(z_1, z_2) = P(z_1)P(z_2)$$

$$\mathcal{D}_{z_1}(\tilde{F})(s, z_2) = J(s)P(z_2), \quad \mathcal{D}_{z_2}(\tilde{F})(s, z_1) = J(s)P(z_1)$$

where  $J$  is a polynomial of the third degree

$$J = -8s^3 + 4(H + 3ac_2^2)s^2 + (8c_2^2b^2 + 2k^2 - 8ac_1^2 - 8c_2^4a^2 - 8c_2^2Ha)s - 8c_1^2b^2 - 4c_2^4ab_4^2c_1^2a^2c_2^2 - k^2c_2^2a - Hk^2 + 2aH^2c_2^2 - 4Hb^2c_2^2 + 4Hc_1^2a + 4c_2^4Ha^2 + 2c_2^6a^3.$$

Finally, as a result of a direct application of Theorem 1, we get

$$\begin{aligned} \frac{d\tilde{s}_1}{\sqrt{\Phi(\tilde{s}_1)}} + \frac{d\tilde{s}_2}{\sqrt{\Phi(\tilde{s}_2)}} &= 0 \\ \frac{\tilde{s}_1 d\tilde{s}_1}{\sqrt{\Phi(\tilde{s}_1)}} + \frac{\tilde{s}_2 d\tilde{s}_2}{\sqrt{\Phi(\tilde{s}_2)}} &= dt, \end{aligned}$$

where

$$\Phi(s) = -4J(s)(s - k)(s + k).$$

□

Notice that the formula (26) has been introduced in [18] together with the variables

$$s_{1,2} = \frac{F(z_1, z_2) \pm \sqrt{P(z_1)P(z_2)}}{2(z_1 - z_2)^2}, \quad (31)$$

and with a claim that the Sokolov system is equivalent to

$$\dot{s}_1 = \frac{\sqrt{P_5(s_1)}}{s_1 - s_2}, \quad \dot{s}_2 = \frac{\sqrt{P_5(s_2)}}{s_2 - s_1}, \quad P_5(s) = P_3(s)P_2(s)$$

with

$$\begin{aligned} P_3(s) &= s(4s^2 + 4sH + H^2 - k^2 + 4c_1^2a + 2c_2^2(Ha - 2b^2) + c_2^4a^2) + 4c_1^2b^2, \\ P_2(s) &= 4s^2 + 4(H + c_2^2a)s + H^2 - k^2 + 2c_2^2ha + c_2^4a^2. \end{aligned}$$

Our variables  $\tilde{s}_i$ , which are the roots of (30) are related with  $s_i$  from [18] in the following manner:

$$\tilde{s}_i = s_i + \frac{H + c_2^2a}{2}.$$

Thus the last Proposition provides a proof of the claim from [18].

### 2.3 A new example of an integrable system of the Kowalevski type

Now, we are going to present a new example of a system of the Kowalevski type. Let us consider the next system of differential equations:

$$\begin{aligned} \dot{p} &= -rq \\ \dot{q} &= -rp - \gamma_3 \\ \dot{r} &= -2q(2p + 1) - 2\gamma_2 \\ \dot{\gamma}_1 &= 2(q\gamma_3 - r\gamma_2) \\ \dot{\gamma}_2 &= 2(p\gamma_3 - r\gamma_1) \\ \dot{\gamma}_3 &= 2(p^2 - q^2)q - 2q\gamma_1 + 2p\gamma_2. \end{aligned} \quad (32)$$

**Lemma 3** *The system (32) preserves the standard measure.*

After a change of variables

$$\begin{aligned} x_1 &= p + q, & e_1 &= x_1^2 + \gamma_1 + \gamma_2, \\ x_2 &= p - q, & e_2 &= x_2^2 + \gamma_1 - \gamma_2, \end{aligned}$$

the system (32) becomes

$$\begin{aligned} \dot{x}_1 &= -rx_1 - \gamma_3 \\ \dot{x}_2 &= rx_2 + \gamma_3 \\ \dot{e}_1 &= -2re_1 \\ \dot{e}_2 &= 2re_2 \\ \dot{r} &= -x_1 + x_2 - e_1 + e_2 \\ \dot{\gamma}_3 &= x_2e_1 - x_1e_2. \end{aligned} \tag{33}$$

The first integrals of the system (33) can be presented in the form

$$\begin{aligned} r^2 &= 2(x_1 + x_2) + e_1 + e_2 + h \\ r\gamma_3 &= -x_1x_2 - x_2e_1 - x_1e_2 - \frac{g_2}{4} \\ \gamma_3^2 &= x_2^2e_1 + x_1^2e_2 - \frac{g_3}{2} \\ e_1 \cdot e_2 &= k^2. \end{aligned} \tag{34}$$

From the integrals (34) we get a relation of the form (16)

$$\begin{aligned} &(x_1 - x_2)^2 e_1 e_2 + \left(2x_1^3 + hx_1^2 - \frac{g_2}{2}x_1 - \frac{g_3}{2}\right) e_2 + \left(2x_2^3 + hx_2^2 - \frac{g_2}{2}x_2 - \frac{g_3}{2}\right) e_1 \\ &- \left(x_1^2x_2^2 + x_1x_2\frac{g_2}{2} + g_3(x_1 + x_2 + \frac{h}{2}) + \frac{g_2^2}{16}\right) = 0. \end{aligned} \tag{35}$$

Without loss of generality, we can assume  $h = 0$  (this can be achieved by a simple linear change of variables  $x_i \mapsto x_i - h/6$ ,  $s \mapsto s - h/6$ ), thus we can use directly the Weierstrass  $\wp$  function. Following the procedure described in Theorem 1 we get

$$\begin{aligned} \frac{dx_1}{\sqrt{P(x_1)}} + \frac{dx_2}{\sqrt{P(x_2)}} &= \frac{ds_1}{\sqrt{P(s_1)}} \\ \frac{dx_1}{\sqrt{P(x_1)}} - \frac{dx_2}{\sqrt{P(x_2)}} &= \frac{ds_2}{\sqrt{P(s_2)}} \end{aligned} \tag{36}$$

where  $P(x)$  denotes the polynomial

$$P(x) = 2x^3 - \frac{g_2}{2}x - \frac{g_3}{2}, \tag{37}$$

and  $s_1, s_2$  are the solutions of quadratic equation in  $s$ :

$$\begin{aligned}
\mathcal{F}(x_1, x_2, s) &:= A(x_1, x_2)s^2 + B(x_1, x_2)s + C(x_1, x_2) \\
&= (x_1 - x_2)^2 s^2 + \left( -2x_1 x_2 (x_1 + x_2) + \frac{g_2}{2}(x_1 + x_2) + g_3 \right) s \\
&\quad + x_1^2 x_2^2 + x_1 x_2 \frac{g_2}{2} + g_3 (x_1 + x_2) + \frac{g_2^2}{16} = 0.
\end{aligned} \tag{38}$$

Finally, we get

**Corollary 1** *The system of differential equations (32) is integrated through the solutions of the system*

$$\begin{aligned}
\frac{ds_1}{\sqrt{\Phi(s_1)}} + \frac{ds_2}{\sqrt{\Phi(s_2)}} &= 0 \\
\frac{s_1 ds_1}{\sqrt{\Phi(s_1)}} + \frac{s_2 ds_2}{\sqrt{\Phi(s_2)}} &= 2 dt,
\end{aligned} \tag{39}$$

where  $\Phi(s) = P(s)(s - k)(s + k)$ .

## 2.4 Explicit integration in genus two theta functions

This subsection is devoted to an explicit integration of the system (32). The integration of the system will be realized in two ways. The first one is based more directly on the Kowalevski original approach [20] and uses the properties of the elliptic functions. The second one follows Kötter's approach (see [19] and Golubev [13]). A generalization of the Kötter transformation was derived in [8] for a polynomial  $P(x)$  of degree four. Here we will reformulate such a transformation for  $P(x)$  of degree three.

We are going to consider here, as in [20], the case where the zeros  $l_i$ ,  $i = 1, 2, 3$  of the polynomial  $P$  of degree three are real and  $l_1 > l_2 > l_3$ . Denote

$$l = (l_1 - l_2)(l_2 - l_3)(l_3 - l_1).$$

Following Kowalevski, we consider the functions

$$P_i = \sqrt{(s_1 - l_i)(s_2 - l_i)}, \quad i = 1, 2, 3 \tag{40}$$

and

$$P_{ij} = P_i P_j \left( \frac{\dot{s}_1}{(s_1 - l_i)(s_1 - l_j)} + \frac{\dot{s}_2}{(s_2 - l_i)(s_2 - l_j)} \right). \tag{41}$$

Then by simple calculations one gets

$$\begin{aligned}
\dot{P}_1 &= \frac{P_3 P_{13} - P_2 P_{12}}{2(l_2 - l_3)}, & \dot{P}_2 &= \frac{P_1 P_{12} - P_3 P_{23}}{2(l_3 - l_1)}, \\
\dot{P}_3 &= \frac{P_2 P_{23} - P_1 P_{13}}{2(l_1 - l_2)}, & \dot{P}_{ij} &= \frac{1}{2} P_i P_j.
\end{aligned} \tag{42}$$

We will now derive the expressions for  $p, q, r, \gamma_1, \gamma_2, \gamma_3$  in terms of  $P_i, P_{ij}$  functions for  $i, j = 1, 2, 3$ .

Denote by

$$du_i = \frac{dx_i}{\sqrt{4x_i^3 - g_2x_i - g_3}}, \quad i = 1, 2.$$

Then  $x_i = \wp(u_i)$ , and we get  $s_1 = \wp(u_1 + u_2), s_2 = \wp(u_1 - u_2)$ .

We will use the following properties of  $\wp$ -function, see [20]:

$$\begin{aligned} \wp(u_1) + \wp(u_2) &= -2 \frac{(l_2^2 - l_3^2)P_1 + (l_3^2 - l_1^2)P_2 + (l_1^2 - l_2^2)P_3}{(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3}, \\ \wp(u_1) - \wp(u_2) &= \frac{-2l}{(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3}, \\ \wp(u_1) \cdot \wp(u_2) &= - \left[ \frac{(l_2 - l_3)(l_1^2 + l_2l_3)P_1 + (l_3 - l_1)(l_2^2 + l_1l_3)P_2}{(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3} \right. \\ &\quad \left. + \frac{(l_1 - l_2)(l_3^2 + l_1l_2)P_3}{(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3} \right]. \end{aligned} \quad (43)$$

After some calculations, we get for the variables  $p, q, r, \gamma_1, \gamma_2, \gamma_3$  the expressions in terms of  $P_i$  and  $P_{ij}$  -functions for  $i, j = 1, 2, 3$ :

$$p = \frac{x_1 + x_2}{2} = \frac{\wp(u_1) + \wp(u_2)}{2} = - \frac{(l_2^2 - l_3^2)P_1 + (l_3^2 - l_1^2)P_2 + (l_1^2 - l_2^2)P_3}{(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3}, \quad (44)$$

$$q = \frac{x_1 - x_2}{2} = \frac{\wp(u_1) - \wp(u_2)}{2} = - \frac{l}{(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3}, \quad (45)$$

$$r = -\frac{\dot{p}}{q} = \frac{1}{2} \frac{(l_1 - l_2)P_{12} + (l_2 - l_3)P_{23} + (l_3 - l_1)P_{13}}{(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3}, \quad (46)$$

$$\begin{aligned} \gamma_1 &= \frac{((l_1 - l_2)P_{12} + (l_2 - l_3)P_{23} + (l_3 - l_1)P_{13})^2}{8((l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3)^2} \\ &\quad - \frac{((l_2^2 - l_3^2)P_1 + (l_3^2 - l_1^2)P_2 + (l_1^2 - l_2^2)P_3)^2 + l^2}{((l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3)^2} \\ &\quad + 2 \frac{(l_2^2 - l_3^2)P_1 + (l_3^2 - l_1^2)P_2 + (l_1^2 - l_2^2)P_3}{(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3}, \end{aligned} \quad (47)$$

$$\begin{aligned}
\gamma_2 &= \frac{l}{((l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3)^2} \\
&\cdot [(l_2 - l_3 - 2l_2^2 + 2l_3^2)P_1 + (l_3 - l_1 - 2l_3^2 + 2l_1^2)P_2 \\
&+ (l_1 - l_2 - 2l_1^2 + 2l_2^2)P_3] \\
&- \frac{((l_2 - l_3)P_2P_3 + (l_3 - l_1)P_1P_3 + (l_1 - l_2)P_1P_2)}{8(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3} \\
&- \frac{(l_2 - l_3)P_{23} + (l_3 - l_1)P_{13} + (l_1 - l_2)P_{12}}{8((l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3)^2} \\
&\cdot (P_3P_{13} + P_1P_{12} + P_2P_{23} - P_2P_{12} - P_1P_{13} - P_3P_{23}), \\
\gamma_3 &= \frac{1}{2} \frac{(l_2^2 - l_3^2)P_1 + (l_3^2 - l_1^2)P_2 + (l_1^2 - l_2^2)P_3}{((l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3)^2} \\
&\cdot ((l_1 - l_2)P_{12} + (l_2 - l_3)P_{23} + (l_3 - l_1)P_{13}) \\
&- \frac{l}{2} \frac{P_3P_{13} + P_1P_{12} + P_2P_{23} - P_2P_{12} - P_1P_{13} - P_3P_{23}}{((l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3)^2}.
\end{aligned} \tag{48}$$

The expressions for  $P_i$  and  $P_{ij}$  for  $i, j = 1, 2, 3$  in terms of the theta-functions are given in [20], [3].

Now, we will perform integration following Kötter [19] and Golubev [13]. First, we will formulate an extension of the Kötter's transformation for a degree three polynomial  $P(x) = 2x^3 - \frac{g_2}{2}x - \frac{g_3}{2}$ .

**Proposition 2** *For a polynomial  $\mathcal{F}(x_1, x_2, s)$  given by the formula (38), there exist polynomials  $\alpha(x_1, x_2, s)$ ,  $\beta(x_1, x_2, s)$ ,  $P(s)$  such that the following identity*

$$\mathcal{F}(x_1, x_2, s) = \alpha^2(x_1, x_2, s) + P(s)\beta(x_1, x_2, s), \tag{50}$$

*is satisfied. The polynomials are defined by the formulae:  $\alpha(x_1, x_2, s) = 2s^2 + s(x_1 + x_2) - x_1x_2 - g_2/4$ ,  $\beta(x_1, x_2, s) = -2(x_1 + x_2 + s)$ , and  $P(s) = 2s^3 - g_2s/2 - g_3/2$ , where  $P$  coincides with the polynomial from formula (37).*

The proof follows by a direct calculation.

Define  $\hat{\mathcal{F}}(s) = \mathcal{F}(x_1, x_2, s)/(x_1 - x_2)^2$ , and consider the identity  $\hat{\mathcal{F}}(s) = (s - u)^2 + (s - u)\hat{\mathcal{F}}'(u) + \hat{\mathcal{F}}(u)$ . Then, from (50) we get

$$\begin{aligned}
&(s - u)^2(x_1 - x_2)^2 + 2(s - u) \left( u(x_1 - x_2)^2 + \frac{B(x_1, x_2)}{2} \right) \\
&+ \alpha^2(x_1, x_2, u) + P(u)\beta(x_1, x_2, u) = 0.
\end{aligned}$$

**Corollary 2** (a) *The solutions  $s_1, s_2$  of the last equation in  $s$  satisfy the following identity in  $u$ :*

$$(s_1 - u)(s_2 - u) = \frac{\alpha^2(x_1, x_2, u)}{(x_1 - x_2)^2} + P(u) \frac{\beta(x_1, x_2, u)}{(x_1 - x_2)^2},$$



where  $P(u)$  is the polynomial defined with (37).

(b) Functions  $P_i$  satisfy

$$P_i = \frac{\alpha(x_1, x_2, l_i)}{x_1 - x_2} = \left(2l_i^2 - \frac{g_2}{4}\right) \frac{1}{x_1 - x_2} + l_i \frac{x_1 + x_2}{x_1 - x_2} - \frac{x_1 x_2}{x_1 - x_2}. \quad (51)$$

Now we introduce a more convenient notation

$$X = \frac{x_1 x_2}{x_1 - x_2}, \quad Y = \frac{1}{x_1 - x_2}, \quad Z = \frac{x_1 + x_2}{x_1 - x_2}.$$

**Lemma 4** *The quantities  $X, Y, Z$  satisfy the system of linear equations*

$$\begin{aligned} -X + \left(2l_1^2 - \frac{g_2}{4}\right) Y + l_1 Z &= P_1 \\ -X + \left(2l_2^2 - \frac{g_2}{4}\right) Y + l_2 Z &= P_2 \\ -X + \left(2l_3^2 - \frac{g_2}{4}\right) Y + l_3 Z &= P_3. \end{aligned} \quad (52)$$

The solutions of the system (52) are

$$\begin{aligned} X &= -\frac{(g_2 + 8l_2 l_3)(l_2 - l_3)P_1 + (g_2 + 8l_3 l_1)(l_3 - l_1)P_2 - (g_2 + 8l_1 l_2)(l_1 - l_2)P_3}{8l}, \\ Y &= \frac{(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3}{-2l}, \\ Z &= \frac{(l_2^2 - l_3^2)P_1 + (l_3^2 - l_1^2)P_2 + (l_1^2 - l_2^2)P_3}{l}. \end{aligned}$$

Using Viète formulae for polynomial  $P(x)$  we can rewrite  $X$  in the form

$$X = \frac{(l_2 - l_3)(l_1^2 + l_2 l_3)P_1 + (l_3 - l_1)(l_2^2 + l_1 l_3)P_2 + (l_1 - l_2)(l_3^2 + l_1 l_2)P_3}{2l}.$$

Now, from the expressions for  $X, Y, Z$  we get  $q = (x_1 - x_2)/2 = 1/(2Y)$ ,  $p = (x_1 + x_2)/2 = Z/(2Y)$ .

The expressions for  $r$  and  $\gamma_i$ ,  $i = 1, 2, 3$  can now be derived in terms of  $P_i, P_{ij}$ -functions from the equations of the system (32), see formulae (44)-(49).

## 2.5 Another method for obtaining systems of Kowalevski type

In [10] a method for constructing examples of systems of the Kowalevski type has been presented.

Now, we will show another method for construction of systems of Kowalevski type, that reduces to (14), (15), (16), with possible first integrals of the form

$$\begin{aligned}
r^2 &= E + p_2 e_1 + p_1 e_2 \\
r\gamma_3 &= F - q_2 e_1 - q_1 e_2 \\
\gamma_3^2 &= G + r_2 e_1 + r_1 e_2 \\
e_1 \cdot e_2 &= k^2.
\end{aligned} \tag{53}$$

We search for functions  $E, F, G, p_i, q_i, r_i$ ,  $i = 1, 2$  starting from

$$(E + p_2 e_1 + p_1 e_2)(G + r_2 e_1 + r_1 e_2) - (F - q_2 e_1 - q_1 e_2)^2 = 0.$$

We want to end up with a relation of the form (16). One set of conditions is the annulation of the coefficients with  $e_1^2, e_2^2$ :

$$p_1 r_1 = q_1^2, \quad p_2 r_2 = q_2^2. \tag{54}$$

We come to the relation

$$\tilde{P}_2 e_1 + \tilde{P}_1 e_2 = C(x_1, x_2) - e_1 e_2 A(x_1, x_2) \tag{55}$$

where

$$A = p_1 r_2 + p_2 r_1 - 2q_1 q_2, \quad C = F^2 - EG, \tag{56}$$

and

$$\tilde{P}_1 = r_1 E + 2q_1 F + p_1 G, \quad \tilde{P}_2 = r_2 E + 2q_2 F + p_2 G. \tag{57}$$

Let us assume:

$$B_1 = E\sqrt{r_1 r_2} + F(\sqrt{p_1 r_2} + \sqrt{p_2 r_1}) + G\sqrt{p_1 p_2}. \tag{58}$$

Then, we have the following

**Lemma 5** *The functions  $A, B_1, C, \tilde{P}_1, \tilde{P}_2$  defined above, satisfy the identity:*

$$B_1^2 - AC = \tilde{P}_1 \tilde{P}_2.$$

**Lemma 6** *For a system which satisfies the relations (53) and (54)-(57), the functions  $f_i$  defined by*

$$f_i = \sqrt{r_i} r + \sqrt{p_i} \gamma_3, \quad i = 1, 2$$

*satisfy the assumption (15).*

*Proof.* Proof is done by a straightforward calculation.

$$\begin{aligned}
f_1^2 &= r_1 r^2 + 2\sqrt{p_1 r_1} r \gamma_3 + p_1 \gamma_3^2 \\
&= r_1 (E + p_2 e_1 + p_1 e_2) + 2q_1 (F - q_2 e_1 - q_1 e_2) + p_1 (G + r_2 e_1 + r_1 e_2) \\
&= E r_1 + G p_1 + 2q_1 F + e_1 (r_1 p_2 + p_1 r_2 - 2q_1 q_2) + e_2 (r_1 p_1 + p_1 r_1 - 2q_1^2) \\
&= \tilde{P}_1 + e_1 (r_1 p_2 + p_1 r_2 - 2\sqrt{r_1 p_2 r_2 p_1}) = \tilde{P}_1 + e_1 A.
\end{aligned} \tag{59}$$

In the same way, we get  $f_2^2 = \tilde{P}_2 + e_2A$ .  $\square$

Now, let us introduce the second assumption:

$$r_i = x_i^2, p_i = (x_i - a)^2, q_i = x_i(x_i - a), i = 1, 2. \quad (60)$$

In order to get a relation of the form (16) from (55), the last, crucial condition, is  $\tilde{P}_1 = P(x_1)$ ,  $\tilde{P}_2 = P(x_2)$ , which means that the functions  $\tilde{P}_1$  and  $\tilde{P}_2$  are equal to **a polynomial  $P$  of one variable,  $x_1$  in the former and  $x_2$  in the latter case.**

In order to satisfy the last requirement, we need to guess a right form of the functions  $E, F, G$ . Let us, finally, assume:

$$\begin{aligned} E &= -2(x_1 - a)^2(x_2 - a)^2 + C_1 \\ F &= x_1(x_1 - a)(x_2 - a)^2 + x_2(x_2 - a)(x_1 - a)^2 + C_2 \\ G &= -2x_1x_2(x_2 - a)(x_1 - a) + C_3. \end{aligned} \quad (61)$$

**Theorem 2** *The polynomials  $E, F, G, p_i, q_i, r_i$  define a completely integrable system, with*

$$f_i = x_i r + (x_i - a)\gamma_3, \quad i = 1, 2. \quad (62)$$

*The system is explicitly integrated on a pinched genus-two curve in the theta-functions of the generalized Jacobian of an elliptic curve.*

Similar systems appeared in a slightly different context in the works of Appel'rot, Mlodzeevskii, Delone in their study of degenerations of the Kowalevski top (see [1], [22],[6]). In particular, we may construct *Delone-type* solutions of the last system:  $s_1 = 0$ ,  $s_2 = \wp(i(t - t_0)/4)$ .

**Proposition 3** *The equations of motion for the completely integrable system described in Theorem 2, with the functions  $f_i$ ,  $i = 1, 2$  defined by (62) are*

$$\begin{aligned} \dot{x}_1 &= -\frac{i}{2}f_1, & \dot{e}_1 &= -me_1, \\ \dot{x}_2 &= \frac{i}{2}f_2, & \dot{e}_2 &= me_2, \\ \dot{r} &= (x_2 - x_1)(x_2 - a)(x_1 - a)ai + \frac{if_2(x_2 - a)}{2r}e_1 - \frac{if_1(x_1 - a)}{2r}e_2 \\ &\quad - \frac{e_1(x_2 - a)^2 - e_2(x_1 - a)^2}{2r}m, \\ \dot{\gamma}_3 &= \frac{e_2x_1^2 - e_1x_2^2}{2\gamma_3}m - \frac{if_2x_2e_1}{2\gamma_3} - \frac{if_1x_1e_2}{2\gamma_3} \\ &\quad - \frac{i(\gamma_3(a - x_1)(a - x_2) - rx_1x_2)(x_2 - x_1)a}{2\gamma_3}, \end{aligned}$$

with

$$m = i \frac{-(r + \gamma_3)f_1^2 e_2 - (\gamma_3 a(r + \gamma_3) - x_2(\gamma_3^2 + 2r\gamma_3 - r^2))f_2 e_1 - ar(x_1 - x_2)f_1 f_2}{f_2^2 e_1 - f_1^2 e_2}. \quad (63)$$

*Proof.* The system of equations described in Theorem 2 is

$$\begin{aligned} \dot{x}_1 &= -\frac{i}{2}(x_1 r + (x_1 - a)\gamma_3), & \dot{e}_1 &= -m e_1 \\ \dot{x}_2 &= \frac{i}{2}(x_2 r + (x_2 - a)\gamma_3), & \dot{e}_2 &= m e_2, \end{aligned}$$

with the first integrals

$$r^2 = -2(x_1 - a)^2(x_2 - a)^2 + C_1 + (x_2 - a)^2 e_1 + (x_1 - a)^2 e_2, \quad (64)$$

$$r\gamma_3 = x_1(x_1 - a)(x_2 - a)^2 + x_2(x_2 - a)(x_1 - a)^2 + C_2 - x_2(x_2 - a)e_1 - x_1(x_1 - a)e_2, \quad (65)$$

$$\gamma_3^2 = -2x_1 x_2 (x_2 - a)(x_1 - a) + C_3 + x_2^2 e_1 + x_1^2 e_2, \quad (66)$$

$$e_1 e_2 = k^2.$$

By differentiating the first integrals (64) and (66) we get equations for  $\dot{r}$  and  $\dot{\gamma}_3$ . Then, by differentiating the first integral (65) we get

$$\begin{aligned} r\dot{\gamma}_3 + \dot{r}\gamma_3 &= \dot{x}_1(x_1 - a)(x_2 - a)^2 + x_1\dot{x}_1(x_2 - a)^2 + x_1(x_1 - a)2(x_2 - a)\dot{x}_2 \\ &+ \dot{x}_2(x_2 - a)(x_1 - a)^2 + x_2\dot{x}_2(x_1 - a)^2 + x_2(x_2 - a)2(x_1 - a)\dot{x}_1 - x_1\dot{x}_1 e_2 \\ &- \dot{x}_1(x_1 - a)e_2 - m x_1(x_1 - a)e_2 - \dot{x}_2(x_2 - a)e_1 - x_2\dot{x}_2 e_1 + m x_2(x_2 - a)e_1. \end{aligned}$$

By plugging the obtained expressions for  $\dot{r}$  and  $\dot{\gamma}_3$  we get an equation for  $m$  with the solution (63).  $\square$

**Proposition 4** *The system of differential equations defined by Proposition 3 is integrated through the solutions of the system*

$$\begin{aligned} \frac{ds_1}{s_1 \sqrt{\Phi_1(s_1)}} + \frac{ds_2}{s_2 \sqrt{\Phi_1(s_2)}} &= 0 \\ \frac{ds_1}{\sqrt{\Phi_1(s_1)}} + \frac{ds_2}{\sqrt{\Phi_1(s_2)}} &= \frac{i}{2} dt, \end{aligned} \quad (67)$$

where  $\Phi_1(s) = s(s - e_4)(s - e_5)$  is a polynomial of degree 3.

By choosing expressions for  $p_i, r_i, i = 1, 2$  and then finding  $E, F, G$  such that  $\tilde{P}_i = Er_i + 2Fq_i + Gp_i$  be polynomials, we can obtain new examples of systems of the Kowalevski type.

## 2.6 Another class of system of Kowalevski type

In this section we will consider another class of systems of Kowalevski type. We consider a situation analogue to that from the beginning of the Section 2.1. The only difference is that the systems we are going to consider now, reduce to (14), where

$$\begin{aligned} f_1^2 &= P(x_1) - \frac{C}{e_2} \\ f_2^2 &= P(x_2) - \frac{C}{e_1}. \end{aligned} \tag{68}$$

The next Proposition is an analogue of Theorem 1. Thus, the new class of systems also has a striking property of jumping genus in integration procedure.

**Proposition 5** *Given a system which reduces to (14), where*

$$\begin{aligned} f_1^2 &= P(x_1) - \frac{C}{e_2} \\ f_2^2 &= P(x_2) - \frac{C}{e_1} \end{aligned} \tag{69}$$

*and integrals reduce to (16);  $A, C, P$  form a discriminantly separable polynomial  $\mathcal{F}$  given with (13). Then the system is linearized on the Jacobian of the curve*

$$y^2 = J(z)(z - k)(z + k),$$

*where  $J$  is a polynomial factor of the discriminant of  $\mathcal{F}$  as a polynomial in  $x_1$  and  $k$  is a constant such that*

$$e_1 e_2 = k^2.$$

Although the proof is a variation of the proof of the Theorem 1, there are some interesting steps and algebraic transformations we would like to point out.

*Proof.* In the same manner as in Theorem 1, we obtain

$$\begin{aligned} \left( \sqrt{e_1} \sqrt{\frac{P(x_2)}{A}} + \sqrt{e_2} \sqrt{\frac{P(x_1)}{A}} \right)^2 &= (s_1 + k)(s_2 - k) \\ \left( \sqrt{e_1} \sqrt{\frac{P(x_2)}{A}} - \sqrt{e_2} \sqrt{\frac{P(x_1)}{A}} \right)^2 &= (s_1 - k)(s_2 + k), \end{aligned}$$

where  $s_1, s_2$  are the solutions of the quadratic equation

$$\mathcal{F}(x_1, x_2, s) = 0$$

in  $s$ . From the last equations, dividing by  $k = \sqrt{e_1 e_2}$ , we get

$$\begin{aligned} 2\sqrt{\frac{P(x_2)}{e_2 A}} &= \frac{1}{k} \left( \sqrt{(s_1 + k)(s_2 - k)} + \sqrt{(s_1 - k)(s_2 + k)} \right) \\ 2\sqrt{\frac{P(x_1)}{e_1 A}} &= \frac{1}{k} \left( \sqrt{(s_1 + k)(s_2 - k)} - \sqrt{(s_1 - k)(s_2 + k)} \right). \end{aligned}$$

Using  $(s_1 - s_2)^2 = 4 \frac{P(x_1)P(x_2)}{A^2}$ , we get

$$\begin{aligned}
f_1^2 &= P(x_1) - \frac{C(x_1, x_2)}{e_2} = \frac{(s_1 - s_2)^2 A^2}{4P(x_2)} - \frac{C}{e_2} = \frac{A^2}{4P(x_2)} \left[ (s_1 - s_2)^2 - \frac{C}{A} \frac{4P(x_2)}{e_2 A} \right] \\
&= \frac{P(x_1)}{(s_1 - s_2)^2} \left[ (s_1 - s_2)^2 - s_1 s_2 \frac{1}{k^2} \left( \sqrt{(s_1 + k)(s_2 - k)} + \sqrt{(s_1 - k)(s_2 + k)} \right)^2 \right] \\
&= \frac{P(x_1)}{(s_1 - s_2)^2} \left[ s_1^2 - 2s_1 s_2 + s_2^2 - \frac{2s_1 s_2}{k^2} \left( s_1 s_2 - k^2 + \sqrt{(s_1^2 - k^2)(s_2^2 - k^2)} \right) \right] \\
&= \frac{P(x_1)}{k^2 (s_1 - s_2)^2} \left[ k^2 (s_1^2 + s_2^2) - 2s_1^2 s_2^2 - 2s_1 s_2 \sqrt{(s_1^2 - k^2)(s_2^2 - k^2)} \right] \\
&= -\frac{P(x_1)}{k^2 (s_1 - s_2)^2} \left[ s_2 \sqrt{s_1^2 - k^2} + s_1 \sqrt{s_2^2 - k^2} \right]^2.
\end{aligned}$$

Similarly

$$f_2^2 = -\frac{P(x_2)}{k^2 (s_1 - s_2)^2} \left[ s_2 \sqrt{s_1^2 - k^2} - s_1 \sqrt{s_2^2 - k^2} \right]^2.$$

From the last two equations and from the equations of motion, we get

$$\begin{aligned}
2\dot{x}_1 &= \frac{-i\sqrt{P(x_1)}}{k(s_1 - s_2)} \left[ s_2 \sqrt{s_1^2 - k^2} + s_1 \sqrt{s_2^2 - k^2} \right] \\
2\dot{x}_2 &= \frac{-i\sqrt{P(x_2)}}{k(s_1 - s_2)} \left[ s_2 \sqrt{s_1^2 - k^2} - s_1 \sqrt{s_2^2 - k^2} \right],
\end{aligned}$$

and

$$\begin{aligned}
\frac{dx_1}{\sqrt{P(x_1)}} + \frac{dx_2}{\sqrt{P(x_2)}} &= \frac{-is_2 \sqrt{s_1^2 - k^2}}{k(s_1 - s_2)} dt \\
\frac{dx_1}{\sqrt{P(x_1)}} - \frac{dx_2}{\sqrt{P(x_2)}} &= \frac{-is_1 \sqrt{s_2^2 - k^2}}{k(s_1 - s_2)} dt.
\end{aligned}$$

Discriminant separability condition (see Corollary 1 from [8] ) gives

$$\begin{aligned}
\frac{dx_1}{\sqrt{P(x_1)}} + \frac{dx_2}{\sqrt{P(x_2)}} &= \frac{ds_1}{\sqrt{J(s_1)}} \\
\frac{dx_1}{\sqrt{P(x_1)}} - \frac{dx_2}{\sqrt{P(x_2)}} &= -\frac{ds_2}{\sqrt{J(s_2)}}.
\end{aligned} \tag{70}$$

Finally

$$\begin{aligned}
\frac{ds_1}{\sqrt{\Phi(s_1)}} + \frac{ds_2}{\sqrt{\Phi(s_2)}} &= \frac{i}{k} dt \\
\frac{s_1 ds_1}{\sqrt{\Phi(s_1)}} + \frac{s_2 ds_2}{\sqrt{\Phi(s_2)}} &= 0,
\end{aligned} \tag{71}$$

where

$$\Phi(s) = J(s)(s - k)(s + k),$$

is a polynomial of degree up to six.  $\square$

### 3 A deformation of the Kowalevski top

In this Section we are going to derive the explicit solutions in genus two theta-functions of the Jurdjevic elasticae [15] and for similar systems [16], [17]. First, we show that we can get the elasticae from the Kowalevski top by using the simplest gauge transformations of the discriminantly separable polynomials.

Consider a discriminantly separable polynomial

$$\mathcal{F}(x_1, x_2, s) := s^2 A + sB + C$$

where

$$A = (x_1 - x_2)^2, B = -2(Ex_1x_2 + F(x_1 + x_2) + G), C = F^2 - EG. \quad (72)$$

A simple affine gauge transformation  $s \mapsto t + \alpha$  transforms  $\mathcal{F}(x_1, x_2, s)$  into

$$\mathcal{F}_\alpha(x_1, x_2, t) = t^2 A_\alpha + tB_\alpha + C_\alpha,$$

with

$$A_\alpha = A, B_\alpha = B + 2\alpha A, C_\alpha = C + \alpha B + \alpha^2 A. \quad (73)$$

Next, we denote  $F_\alpha = F + \alpha F_1$ ,  $E_\alpha = E + \alpha E_1$ ,  $G_\alpha = G + \alpha G_1$ . From

$$C_\alpha = F_\alpha^2 - E_\alpha G_\alpha,$$

by equating powers of  $\alpha$ , we get

$$B = 2FF_1 - E_1G - EG_1, \quad A = F_1^2 - E_1G_1. \quad (74)$$

From (72) one obtains

$$F_1 = -(x_1 + x_2), G_1 = 2x_1x_2, E_1 = 2. \quad (75)$$

One easily checks that  $F_1^2 - E_1G_1 = A$ ,

$$\begin{aligned} E_\alpha &= 6l_1 - (x_1 + x_2)^2 + 2\alpha \\ F_\alpha &= 2cl + x_1x_2(x_1 + x_2) - \alpha(x_1 + x_2) \\ G_\alpha &= c^2 - k^2 - x_1^2x_2^2 + 2\alpha x_1x_2. \end{aligned} \quad (76)$$

Jurdjevic considered a deformation of the Kowalevski case associated to the Kirchhoff elastic problem, see [15]. The systems are defined by the Hamiltonians

$$H = \frac{1}{4} (M_1^2 + M_2^2 + 2M_3^2) + \gamma_1$$

where the deformed Poisson structures  $\{\cdot, \cdot\}_\tau$  are defined by

$$\{M_i, M_j\}_\tau = \epsilon_{ijk} M_k, \quad \{M_i, \gamma_j\}_\tau = \epsilon_{ijk} \gamma_k, \quad \{\gamma_i, \gamma_j\}_\tau = \tau \epsilon_{ijk} M_k,$$

and where the deformation parameter takes values  $\tau = 0, 1, -1$ . These structures correspond to  $e(3)$ ,  $so(4)$ , and  $so(3, 1)$  respectively. The classical Kowalevski case corresponds to the case  $\tau = 0$ . The systems with  $\tau = -1, 1$  have been considered by St. Petersburg's school (Komarov, Kuznetsov [17]) in 1990's, and they have been rediscovered by several authors in the meantime. Here, we are giving explicit formulae in theta-functions for the solutions of these systems.

Denote

$$\begin{aligned} e_1 &= x_1^2 - (\gamma_1 + i\gamma_2) + \tau \\ e_2 &= x_2^2 - (\gamma_1 - i\gamma_2) + \tau, \end{aligned}$$

where

$$x_{1,2} = \frac{M_1 \pm iM_2}{2}.$$

The integrals of motion

$$\begin{aligned} I_1 &= e_1 e_2 \\ I_2 &= H \\ I_3 &= \gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3 \\ I_4 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2 + \tau(M_1^2 + M_2^2 + M_3^2) \end{aligned}$$

may be rewritten in the form (53)

$$\begin{aligned} k^2 &= I_1 = e_1 \cdot e_2 \\ M_3^2 &= e_1 + e_2 + \hat{E}(x_1, x_2) \\ -M_3 \gamma_3 &= -x_2 e_1 - x_1 e_2 + \hat{F}(x_1, x_2) \\ \gamma_3^2 &= x_2^2 e_1 + x_1^2 e_2 + \hat{G}(x_1, x_2), \end{aligned}$$

where

$$\begin{aligned} \hat{G}(x_1, x_2) &= -x_1^2 x_2^2 - 2\tau x_1 x_2 - 2\tau I_2 + \tau^2 + I_4 - I_1 \\ \hat{F}(x_1, x_2) &= (x_1 x_2 + \tau)(x_1 + x_2) - I_3 \\ \hat{E}(x_1, x_2) &= -(x_1 + x_2)^2 + 2(I_2 - \tau). \end{aligned}$$

**Proposition 6** *An affine gauge transformation  $s \mapsto t + \alpha$  transforms the Kowalevski top with  $c = -1$  to the Jurdjevic elasticae according to the formulae*

$$\tau = -\alpha, \quad I_2 = 3l_1, \quad I_3 = 2l, \quad I_4 = 1 - \alpha^2 - 6l_1\alpha.$$

We can apply the generalized Kötter transformation derived in [8] to obtain the expressions for  $M_i, \gamma_i$  in terms of  $P_i$  and  $P_{ij}$  functions for  $i, j = 1, 2, 3$ . First we



will rewrite the equations of motion for Jurdjevic elasticae:

$$\begin{aligned}
\dot{M}_1 &= \frac{M_2 M_3}{2} \\
\dot{M}_2 &= -\frac{M_1 M_3}{2} + \gamma_3 \\
\dot{M}_3 &= -\gamma_2 \\
\dot{\gamma}_1 &= -\frac{M_2 \gamma_3}{2} + M_3 \gamma_2 \\
\dot{\gamma}_2 &= \frac{M_1 \gamma_2}{2} - M_3 \gamma_1 + \tau M_3 \\
\dot{\gamma}_3 &= -\frac{M_1 \gamma_2}{2} + \frac{M_2 \gamma_1}{2} - \tau M_2.
\end{aligned} \tag{77}$$

Now we introduce the following notation:

$$\begin{aligned}
R(x_1, x_2) &= \hat{E}x_1x_2 + \hat{F}(x_1 + x_2) + \hat{G}, \\
R_1(x_1, x_2) &= \hat{E}\hat{G} - \hat{F}^2, \\
P(x_i) &= \hat{E}x_i^2 + 2\hat{F}x_i + \hat{G}, \quad i = 1, 2.
\end{aligned}$$

**Lemma 7** *For polynomial  $\mathcal{F}(x_1, x_2, s)$  given with*

$$\mathcal{F}(x_1, x_2, s) = (x_1 - x_2)^2 s^2 - 2R(x_1, x_2)s - R_1(x_1, x_2),$$

*there exist polynomials  $A(x_1, x_2, s)$ ,  $B(x_1, x_2, s)$ ,  $f(s)$ ,  $A_0(s)$  such that the following identity holds*

$$\mathcal{F}(x_1, x_2, s)A_0(s) = A^2(x_1, x_2, s) + f(s)B(x_1, x_2, s). \tag{78}$$

*The polynomials are defined by the formulae:*

$$\begin{aligned}
A_0(s) &= 2s + 2I_1 - 2\tau \\
f(s) &= 2s^3 + 2(I_1 - 3\tau)s^2 + (-4\tau(I_1 - \tau) - 2I_2 + 4\tau^2 + 2I_4 - 4\tau I_2)s \\
&\quad + (I_1 - \tau)(-2I_1 + 2\tau^2 + 2I_4 - 4\tau I_2) - I_3^2 + 2(I_1 - \tau)\tau^2 \\
A(x_1, x_2, s) &= A_0(s)(x_1x_2 - s) - I_3(x_1 + x_2) + 2\tau(I_1 - \tau) + 2\tau s \\
B(x_1, x_2, s) &= (x_1 + x_2)^2 - 2s - 2I_1 + 2\tau.
\end{aligned}$$

Denote by  $m_i$  the zeros of polynomial  $f$  and

$$P_i = \sqrt{(s_1 - m_i)(s_2 - m_i)} \quad i = 1, 2, 3.$$

The same way as in Corollary 2 we get

$$P_i = \frac{\sqrt{A_0(m_i)}(x_1x_2 - m_i)}{x_1 - x_2} + \frac{-I_3(x_1 + x_2) + 2\tau(I_1 - \tau + m_i)}{(x_1 - x_2)\sqrt{A_0(m_i)}}, \quad i = 1, 2, 3. \tag{79}$$

Put

$$\begin{aligned} X &= \frac{x_1 x_2}{x_1 - x_2}, \quad Y = \frac{1}{x_1 - x_2}, \\ Z &= \frac{-I_3(x_1 + x_2) + 2\tau(I_1 - \tau)}{x_1 - x_2}, \\ n_i &= A_0(m_i) = 2m_i + 2I_1 - 2\tau, \quad i = 1, 2, 3. \end{aligned}$$

The relations (79) can be rewritten as a system of linear equations

$$\begin{aligned} X + Y m_1 \left( \frac{2\tau}{n_1} - 1 \right) + \frac{Z}{n_1} &= \frac{P_1}{\sqrt{n_1}} \\ X + Y m_2 \left( \frac{2\tau}{n_2} - 1 \right) + \frac{Z}{n_2} &= \frac{P_2}{\sqrt{n_2}} \\ X + Y m_3 \left( \frac{2\tau}{n_3} - 1 \right) + \frac{Z}{n_3} &= \frac{P_3}{\sqrt{n_3}}. \end{aligned}$$

The solutions of the previous system are

$$\begin{aligned} Y &= - \sum_{i=1}^3 \frac{\sqrt{n_i} P_i}{f'(m_i)} \\ X &= - \sum_{i=1}^3 \frac{P_i \sqrt{n_i}}{f'(m_i)} (m_j + m_k + I_1 - 2\tau) \\ Z &= \sum_{i=1}^3 \frac{2\sqrt{n_i} P_i}{f'(m_i)} \left( \frac{n_j \cdot n_k}{4} + \tau(\tau - I_1) \right), \end{aligned} \tag{80}$$

with  $(i, j, k)$  - a cyclic permutation of  $(1, 2, 3)$ .

Finally, we obtain

**Proposition 7** *The solutions of the system of differential equations (77) in terms of  $P_i$ ,  $P_{ij}$  functions are given with*

$$\begin{aligned} M_1 &= \frac{\sum_{i=1}^3 \frac{2\sqrt{n_i} P_i}{f'(m_i)} \left( \frac{n_j \cdot n_k}{4} + \tau(\tau - I_1) \right)}{I_3 \sum_{i=1}^3 \frac{\sqrt{n_i} P_i}{f'(m_i)}} + \frac{2\tau(I_1 - \tau)}{I_3} \\ M_2 &= - \frac{1}{i \sum_{i=1}^3 \frac{\sqrt{n_i} P_i}{f'(m_i)}} \\ M_3 &= \frac{2i \sum_{k=1}^3 \frac{n_k \sqrt{n_i n_j} P_{ij}}{f'(m_k)}}{\sum_{i=1}^3 \frac{\sqrt{n_i} P_i}{f'(m_i)}} \end{aligned}$$

and

$$\begin{aligned}
\gamma_1 &= I_2 + \frac{1}{8} \left( \frac{\sum_{k=1}^3 \frac{n_k \sqrt{n_i n_j} P_{ij}}{f'(m_k)}}{\sum_{i=1}^3 \frac{\sqrt{n_i} P_i}{f'(m_i)}} \right)^2 - \frac{\sum_{i=1}^3 \frac{P_i \sqrt{n_i}}{f'(m_i)} (m_j + m_k + I_1 - 2\tau)}{\sum_{i=1}^3 \frac{P_i \sqrt{n_i}}{f'(m_i)}} \\
\gamma_2 &= -2i \frac{(\sum_{k=1}^3 \frac{n_k \sqrt{n_i n_j}}{f'(m_k)} \frac{P_i P_j}{2}) \cdot (\sum_{i=1}^3 \frac{\sqrt{n_i} P_i}{f'(m_i)})}{\left( \sum_{i=1}^3 \frac{\sqrt{n_i} P_i}{f'(m_i)} \right)^2} \\
&\quad + 2i \frac{(\sum_{k=1}^3 \frac{n_k \sqrt{n_i n_j} P_{ij}}{f'(m_k)}) \cdot (\sum_{i=1}^3 \frac{\sqrt{n_i}}{f'(m_i)} \frac{P_k P_{ik} - P_j P_{ij}}{2(m_j - m_k)})}{\left( \sum_{i=1}^3 \frac{\sqrt{n_i} P_i}{f'(m_i)} \right)^2} \\
\gamma_3 &= \frac{\sum_{k=1}^3 \frac{\sqrt{n_i n_j} P_{ij}}{f'(m_k)}}{2i \sum_{i=1}^3 \frac{\sqrt{n_i} P_i}{f'(m_i)}}.
\end{aligned}$$

As we mentioned before, the formulae expressing  $P_i, P_{ij}$  in terms of the theta-functions are given [20]. This gives explicit formulae for the elasticae.

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